# A Study on the Construction of Hybrid Type Four Sided Elements and Exact Calculations of Element Matrices

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**Abstract**

This paper is concentrated to form the exact element matrices for four sided finite elements. The construction of higher order hybrid type four sided elements with their exact matrices in terms of algebraic expressions is also focused. The matrices are formed in such a way that no integration is required. The numerical integration scheme, such as Gaussian quadrature takes large number of Gaussian points for higher accuracy. But more number of Gaussian points makes the scheme inefficient. Hence to make a proper balance between accuracy and efficiency, the elements of simple shapes for which the components of matrices are exactly known, are used by well known assembly process to form the desired matrices for four sided elements. Hence, algebraic expressions in terms of nodal coordinates are presented and for their computation, a code is developed. Several examples available in the literatures are considered to test the exactness and efficiency of the calculation of element matrices with these developed expressions. In all cases, it is observed that the present technique is faster and exact. Thus, we believe that this study will contribute directly to the finite element method solution procedure for continuum mechanics problems to obtain desired accuracy in the solutions.

*Kyewords:* hybrid type four sided elements, element matrices, finite element method.

## 1. Introduction

It is well known that the square domain integrals are frequently encountered in fields of sciences and engineering and most of them defies our analytical skills and we are resort to numerical integration schemes. Further, among all the numerical integration methods, the Gaussian quadrature rules are more attractive and hence widely used. Particularly in finite element method, for solving boundary value problems the Gaussian quadrature rules are being used unanimously. Because in this case most of the integrals encountered as the rational integrals and for which analytical formulae are not yet established. For such evaluations of the integrals necessary Gauss points and weights are increased to improve the accuracy and that require much computing time. Hence, an important task arises for a proper balance between accuracy and efficiency. With this view, a technique is developed here to evaluate all the square domain integrals with higher accuracy by efficiently reducing the computing time. It was possible through division of the original square domain into two triangular domain and each triangular domain integrals are then evaluated by using lower order Gaussian quadrature rules. A complete program in FORTRAN compatible with the formulations is developed which can be used to evaluate all the square domain integrals easily encountered in the realm of science and engineering.

## 2. Development of the technique using two subdivisions

Consider the following integral for evaluation.

$$I = \int\int_{\mu} f(x,y)dxdy \tag{1}$$

The domain μ of the integral is shown in the Fig. 1. The entire domain μ is splitted into two linear triangles $e_1$ and $e_2$.

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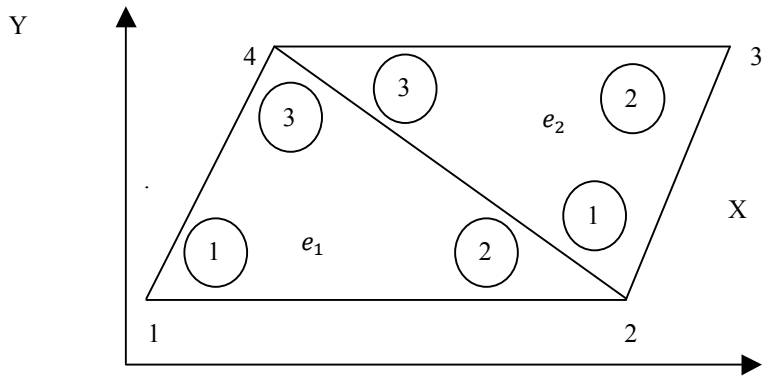



Fig. 1. Original domain μ and its two subdivisions.

Relation between local and global nodes:

Table 1. Relation between local and global nodes

| Local nodes | Global nodes($e_1$) | Global nodes($e_2$) |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 3 |
| 3 | 4 | 4 |

By the property of the definite integral, we can write,

$$I = \int\int_{\mu} f(x,y)dxdy = \int\int_{e_1} f(x,y)dxdy + \int\int_{e_2} f(x,y)dxdy = I_1 + I_2 \quad (2)$$

where, $I_1 = \int\int_{e_1} f(x,y)dxdy$ (3)

and $I_2 = \int\int_{e_2} f(x,y)dxdy$ (4)

The parametric transformation equations, which transforms the triangle in $(x,y)$ space into standard triangle in $(s,t)$ space is given by [1,2,3],

$$x = x_1N_1 + x_2N_2 + x_3N_3 = \frac{1}{2}[(x_2 - x_1)s + (x_3 - x_1)t + (x_2 + x_3)] \quad (5)$$

$$y = y_1N_1 + y_2N_2 + y_3N_3 = \frac{1}{2}[(y_2 - y_1)s + (y_3 - y_1)t + (y_2 + y_3)] \quad (6)$$

Where, the shape functions $N_i(s,t), i = 1,2\ \&3$ are given by, $N_1 = \left(-\frac{1}{2}\right)(s+t)$, $N_2 = \left(\frac{1}{2}\right)(s+1)$ and $N_3 = \left(\frac{1}{2}\right)(t+1)$.

Then the Jacobian transformation becomes

$$J\left(\frac{x,y}{s,t}\right) = \frac{1}{4}\{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\} \quad (7)$$

For the element $e_1$: From Equations (5) & (6) we get,

$$x = \frac{1}{2}[(x_2 - x_1)s + (x_4 - x_1)t + (x_2 + x_4)] = XX1 \quad (8)$$

$$y = \frac{1}{2}[(y_2 - y_1)s + (y_4 - y_1)t + (y_5 + y_4)] = YY1 \quad (9)$$

Hence, the Jacobian transformation for element $e_1$ becomes,

$$J_1 = \frac{1}{4}\{(x_2 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_2 - y_4)\}$$

Hence, $I_1 = \int_{s=-1}^{1}\int_{t=1}^{s} f(XX1, YY1)J_1 dsdt$

For the element $e_2$: From Equation (5) & (6) we get,

$x = \frac{1}{2}[(x_3 - x_2)s + (x_4 - x_2)t + (x_3 + x_4)] = XX2$

$y = \frac{1}{2}[(y_3 - y_2)s + (y_4 - y_2)t + (y_3 + y_4)] = YY2$

Hence, the Jacobian transformation for element $e_2$ becomes,

$J_2 = \frac{1}{4}\{(x_3 - x_2)(y_4 - y_2) - (x_4 - x_2)(y_3 - y_2)\}$

Hence, $I_2 = \int_{s=-1}^{1}\int_{t=1}^{s} f(XX2, YY2)J_2 dsdt$

Now using the transformation $s = \xi$ and $t = \frac{1}{2}(-1 - \xi + \eta - \xi\eta)$, we have the Jacobian of the new transformation $J_3 = \frac{1}{2}(1 - \xi)$ and then the Gaussian quadrature formula for triangle to evaluate the integral in Eqs.(5) & (6) are [1,2,3] ,

$I_1 = \int_{-1}^{1}\int_{-1}^{1} f_1\left(\xi, \frac{1}{2}(-1 - \xi + \eta - \xi\eta)\right) J_1 J_3 \, d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1} g_1(\xi, \eta)\, d\xi d\eta = \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j g_1(\xi_i, \eta_j)$

$I_2 = \int_{-1}^{1}\int_{-1}^{1} f_2\left(\xi, \frac{1}{2}(-1 - \xi + \eta - \xi\eta)\right) J_2 J_3 \, d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1} g_2(\xi, \eta)\, d\xi d\eta = \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j g_2(\xi_i, \eta_j)$

Now, we have, $I = \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j g_1(\xi_i, \eta_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j g_2(\xi_i, \eta_j)$

Where $(\xi_i, \eta_j)$ are the ij-th sampling points, $w_i$ , $w_j$ are weights and $m, n$ denote the number of gauss points in the formula. A program in FORTRAN is developed to check the accuracy and the efficiency of the technique.

**3. Application Examples**

Exampe-1: The exact value of the integral $F(x, y) = \int_{X=1}^{2}\int_{Y=0}^{1} e^x e^y dydx$ is 8.025706554. Following table is obtained by 2 points to 10 points Gaussian quadrature rules employed by the new technique developed in this paper using two triangular elements.

Table 1. Integral value for two triangular elements from example 1.

| Number of points | Integral value |
|---|---|
| 2 | 7.992782334 |
| 3 | 8.025434729 |
| 4 | 8.025705301 |
| 5 | 8.025706528 |
| 6 | 8.025706603 |
| 7 | 8.025706549 |
| 8 | 8.025706553 |
| 9 | 8.025706590 |
| 10 | 8.025706520 |

Exampe-2: The exact value of the integral $F(x, y) = \int_{X=1}^{2}\int_{Y=2}^{3} \frac{1}{y} dydx$ is 0.69314718. Following table is obtained by 2 points to 10 points Gaussian quadrature rules employed by the new technique developed in this paper using two triangular elements.

Table 2. Integral value for two triangular elements from example 2.

| Number of points | Integral value |
|---|---|
| 2 | 0.6941091755 |
| 3 | 0.6931745693 |
| 4 | 0.6931479811 |
| 5 | 0.6931472004 |
| 6 | 0.6931471845 |
| 7 | 0.6931471884 |
| 8 | 0.6931471807 |
| 9 | 0.6931471853 |
| 10 | 0.6931471853 |

Exampe-3: The exact value of the integral $F(x,y) = \int_{X=1}^{2}\int_{Y=2}^{3} x^2 e^{x^3} dydx$ is .572760609. Following table shows the integral value of $F(x,y)$ obtained by 2 points to 10 points Gaussian quadrature rules employed by the new technique developed in this paper using two triangular elements.

Table 3. Integral value for two triangular elements from example 3.

| Number of points | Integral value |
|---|---|
| 2 | 0.5094860545 |
| 3 | 0.5667565110 |
| 4 | 0.5723543550 |
| 5 | 0.5727390965 |
| 6 | 0.5727596587 |
| 7 | 0.5727605767 |
| 8 | 0.5727606009 |
| 9 | 0.5727606095 |
| 10 | 0.5727606219 |

## 4. Conclusion

A new technique is developed to evaluate the square domain integrals with higher efficiency and accuracy. This technique is fully based on geometric transformations and hence it is correct and stable. Explicitly, the technique comprises two ways to integrate the original integrals employing lower order Gaussian quadrature rules. Some test examples are considered to substantiate the efficiency and accuracy of the developed technique. A complete FORTRAN program is also developed which can be used as required in as many fields. The computed results clearly show the efficiency and the accuracy of the present technique and the program will find better application in numerical methods as well as in science and engineering.

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